Geometry from Entanglement

Charles Rabideau

Vrije Universiteit Brussel / University of Pennsylvania

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1705.03026 w/ T. Faulkner, F. Haehl, E. Hijano, O. Parrikar and M. Van Raamsdonk
1712.06620 w/ F. Haehl, E. Hijano and O. Parrikar

1812.06985 w/ V. Balasubramanian

Information theory provides us with a set of quantities and tools which directly access the geometry of holographic theories of gravity.

This allows us to address many important questions in holography in terms of information theory.

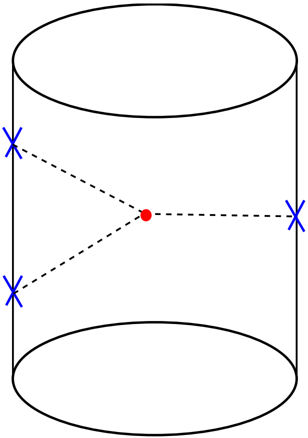
- What controls the dynamics of holographic spacetimes?
- Are there constraints on the matter content of UV complete quantum gravity?
- Can we propose new quantities of relevance to information theory inspired by geometry?

Outline

Quantum gravity in a spacetime with given asymptotic boundary conditions is dual to a quantum field theory living on the boundary.

Operators in the CFT are dual to fields in the bulk. The stress tensor is dual to fluctuations of the bulk metric.

$\mathrm{AdS}_{d+1} \leftrightarrow \mathrm{CFT}_d$:

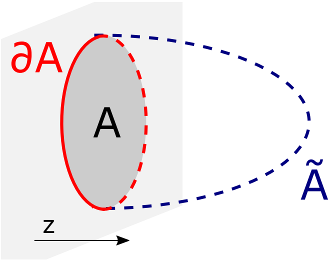


Hubeny-Rangamani-Ryu-Takayanagi (HRRT) Formula

$$S_A = \frac{1}{4G_N}\mathrm{Area}(\widetilde{A})$$

$\widetilde{A}$ is:

- extremal-area
- codimension-two
- spacelike
- homologous to A
- of minimum area



Outline

How far we can push a slogan like "Geometry comes from entanglement"?

What encodes the dynamics of holographic spacetimes?

Can we identify necessary conditions for the existence of a dual geometry?

Spacetimes which compute the entanglement entropy of a field theory via the HRRT formula must obey Einstein's equations.

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We will proceed in an expansion around empty AdS.

The linearised Einstein's equations can be derived directly from a "First Law of Entanglement" analogous to the laws of thermodynamics.[1]

We then developed a framework to extend this result order by order.[2] We worked out all the details at second order. Higher orders get progressively more difficult, but we were able to identify an overarching structure to the framework.

[1] Lashkari et al. [1308.3716] and Faulkner et al. [1312.7856]

[2] [1705.03026, 1712.06620] w/ T. Faulkner, F. Haehl, E. Hijano, O. Parrikar and M. Van Raamsdonk

Outline

1. Holography Primer
2. Einstein's Equations from Entanglement
   - Goal
   - Relative Entropy
   - Deriving Einstein's Equations
3. Differential Entropy
   - Geometry from Information

The main information theoretic tool which we will use is the relative entropy, a close cousin to the entanglement entropy but that is well defined in field theory.

It has a holographic interpretation in terms of a bulk quasi-local energy.

Information theory describes constraints that the relative entropy must obey, in particular it is positive.

By computing it in field theory, we will be able to conclude that holographic spacetimes obey Einstein's equations.

Relative entropy is a measure of the distinguishability of two states.

Given the state of a subsystem of a field theory, we can ask how difficult it would be for an observer confined to a subregion to distinguish it from a given reference state, often the vacuum.

Relative entropy

$$S(\rho_A||\sigma_A) = \mathrm{tr}(\rho_A \log(\rho_A)) - \mathrm{tr}(\rho_A \log(\sigma_A))$$
$$= -\Delta S_A + \Delta\langle H_A\rangle$$

Relative entropy is a generalisation of free energy

$$\sigma = e^{-\beta H} \implies S(\rho||\sigma) = \beta F = \beta E - S$$

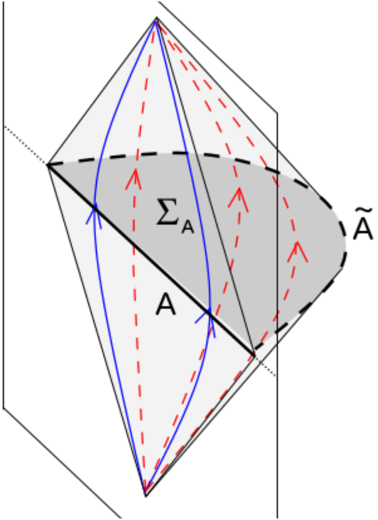
Modular Hamiltonian

$$H_A = -\log(\sigma_A)$$

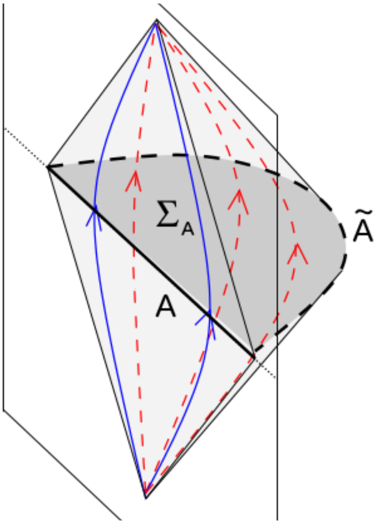
Modular Flow:

For a ball shaped region in the vacuum of a CFT, the modular flow is local and geometric.

It is associated to a bulk Killing vector of AdS which preserves the bulk entanglement wedge.



The bulk Hamiltonian generating translations along the bulk Killing vector ξ_A, can be re-expressed as a quantity localised on the boundary of Σ_A, up to a term involving the equations of motion.



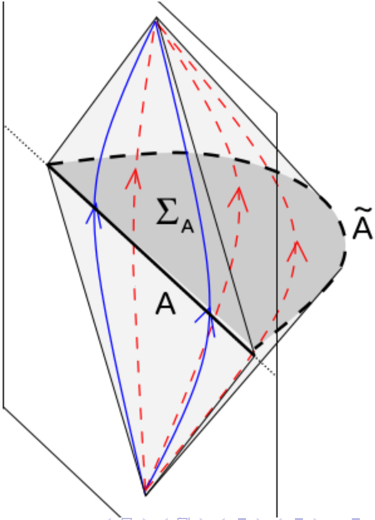
Hollands-Wald

The boundary quantity on A is the Hamiltonian generating the flow along ζ_A, which is the vacuum modular Hamiltonian.

The term supported on $\tilde{A}$ is the area S_A.

This is dual to relative entropy!

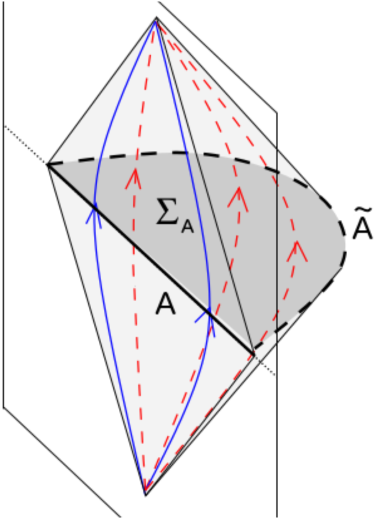
$$S_{rel} = \langle H \rangle - S$$



$$\delta S_{rel} = \delta\left(E_{\zeta_A} - Area(\tilde{A})\right)$$
$$= \delta(\Omega_{\xi_A}) + EOM_{\Sigma_A}(\delta g)$$

The relative entropy is positive, therefore, when the equations of motion are obeyed, the quasi-local notion of energy Ω_{ξ_A} must be positive as well.

It is also monotonic leading to further constraints.



[1412.3514] w/ N. Lashkari, P. Sabella-Garnier and M. Van Raamsdonk, [1605.01075] Lashkari et al.

Outline

$$\delta S_{rel} = \delta(\Omega_{\xi_A}) + EOM_{\Sigma_A}(\delta g)$$

This identity includes a term which vanishes when Einstein's equations are satisfied.

By computing the relative entropy and rewriting it in terms of the canonical energy of a putative dual spacetime, we show that the term involving the equations of motion must vanish. This implies that the equations of motion must hold.

We will work order by order in an expansion around empty AdS, where we know the modular Hamiltonian of the reference state.

To leading order the matching is trivial: the first law of entanglement entropies ensures that the variation of the relative entropy vanishes.

Laws of entanglement entropy

$$S(\rho||\sigma) = \langle H \rangle - S$$

Relative entropy reduces to free energy

$$\sigma = e^{-\beta H} \implies S(\rho||\sigma) = \beta F = \beta E - S$$

Similarly, the laws of thermodynamics are simply the "laws of entanglement entropy" applied to the case where the reference state is a thermal ensemble.

First law of entanglement:

$$\delta S = \delta \langle H \rangle$$

This establishes the result[3] that linearised Einstein's equations follow from HRRT and the first law.

For any given CFT, we can construct an auxiliary dual spacetime which computes the entanglement entropy of ball shaped regions and the one-point function of single trace operators. Any such spacetime must obey Einstein's equations.

[3] Lashkari et al. [1308.3716] and Faulkner et al. [1312.7856]

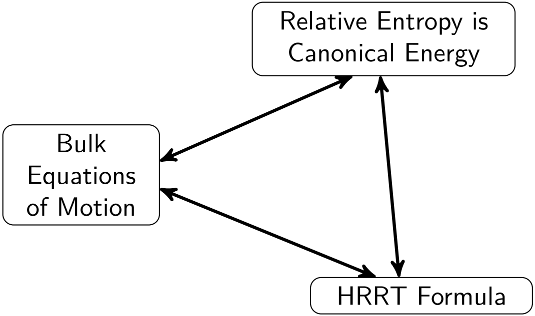
To go to higher orders we need to know more and more about the CFT in order to rewrite the relative entropy in terms of the canonical energy.

This identifies conditions that a CFT must obey in order to have a gravitational dual obeying Einstein's equations.

At leading order, we didn't need to impose any conditions on the CFT: universality ensures that we can construct an auxiliary dual space obeying Einstein's equations for any CFT.

The next order imposes conditions on a few generalised central charges. At higher orders, non-universal information appears.

This formalism leads to a web of relations, where any two can be used to derive the third.



Recap:

Information theoretic tools were useful in deriving a constraint on the dynamics of a dual spacetime.

By computing the relative entropy, we were able to conclude:

Spacetimes which compute the entanglement entropy of a field theory via the HRRT formula must obey Einstein's equations.

We also saw that constraints on the relative entropy can be rephrased in terms of bulk energy conditions.

Outline

We've seen that the entanglement entropy of in a field theory is dual to the area of a minimal surface. What is the dual of the area of a non-minimal surface?

The differential entropy: a measure of the information inaccessible to a family of observers making measurements limited in time.

Consider a family of observers at locations x_i each making observations for a time $T(x_i)$. These have access to a subset of the bulk spacetime.

This region is bounded by a non-extremal surface.

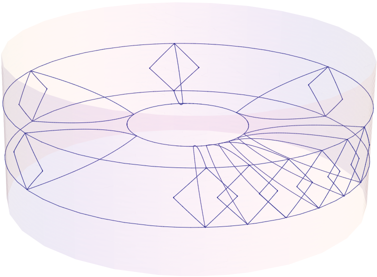


Figure credit: Balasubramanian et al. [1310.4204]

In AdS_3, it can be shown that a given combination of entanglement entropies constructed from the regions accessible to these observers, known as the *differential entropy* computes the area of this non-extremal surface.

This differential entropy can be understood as the entanglement cost of reconstructing the joint state of a number of subsystems from the state of each subsystem.

$$S_{diff} = \sum_{i=1}^{N} \Big[S(I_i) - S(I_i \cap I_{i-1}) \Big] \to \int du \Big[\partial_v S([u, v]) \Big]_{v=v(u)}$$

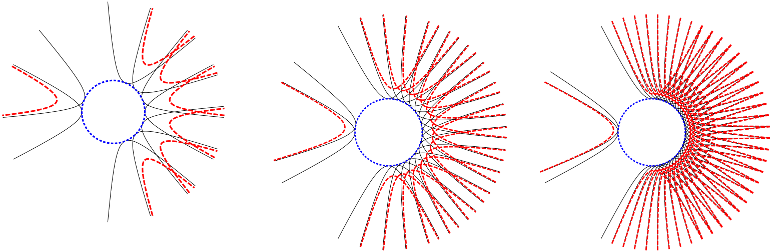


Figure credit: Balasubramanian et al. [1310.4204]

We recently proposed an extension of this concept to higher dimensions, but a general proof of its applicability is still a work in progress.

It has the form of a particular shape derivative of the entanglement entropy:

$$S_{diff} \sim \frac{\delta^{d-1} S}{\delta A_1 \ldots \delta A_{d-1}}$$

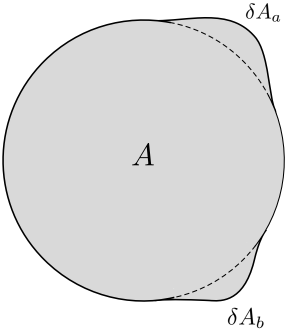


Figure credit: Faulkner, Leigh, Parrikar. [1511.05179]

From the gravitational point of view, these non-extremal surfaces obey certain area laws. For example, if every observer gets to make measurements for strictly less time, the area of the non-extremal surface must decrease.

This indicates that the notion of differential entropy, originally proposed for geometric reasons, is an interesting information theoretic quantity of its own right.

Summary

We have seen a number of ways in which the connection between information theory and geometry have enriched both subjects.

Constraints on the information in a field theory, can be used to constrain the energy and dynamics of any potential dual theory.

Notions developed to better understand the geometry of holographic duals also lead to new information theoretic quantities of interest.